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APPLYING THE RECIPROCITY PRINCIPLE TO SOLVING HEAT CONDUCTIVITY PROBLEMS

A.I. Pekhovich

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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER Draft Translation 599	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) APPLYING THE RECIPROCITY PRINCIPLE TO SOLVING HEAT CONDUCTIVITY PROBLEMS		5. TYPE OF REPORT & PERIOD COVERED
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) A.I. Pekhovich		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS U.S. Army Cold Regions Research and Engineering Laboratory Hanover, New Hampshire		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS		12. REPORT DATE February 1977
		13. NUMBER OF PAGES 16
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report)
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

See next page

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

HEAT TRANSFER

Delta t

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

It has been shown that the phenomenon of reciprocity which is known in physics also occurs in the case of non-established heat propagation in a solid, specifically if the heat source is located at point 1 and at point 2 causes a temperature change Δt, then when the source is shifted to point 2 at point 1 there occurs the same change in temperature Δt. This makes it possible to obtain analytical solutions to some already known problems, as well as to new problems. The application of the reciprocity principle

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appears promising in summing up laboratory and full-scale studies.

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DRAFT TRANSLATION 599

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ENGLISH TITLE: APPLYING THE RECIPROCITY PRINCIPLE TO SOLVING HEAT
CONDUCTIVITY PROBLEMS

FOREIGN TITLE: (PRILPZHENIYE PRINTSIPA VZAIMNOSTI K PESHENIYU ZADACH
TEPLOPROVODNOSTI),

AUTHOR: 10 A.I. Pekhovich

11 Feb 77   12 20p

14 CRREL-TL-599

21 Trans. of

Translated by Office of the Assistant Chief of Staff for Intelligence for
U.S. Army Cold Regions Research and Engineering Laboratory, 1977, 16p.

NOTICE

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Designations

x -- coordinate; τ -- time; h -- thickness of plate; t -- thickness of body; t_0 -- initial temperature of body; Δt -- change in temperature of body; i_s -- heat source of given power (heat source); I_t -- heat source of given temperature (temperature source); ϑ -- ambient temperature; α -- heat emission factor; S -- density of heat flow; λ -- thermal conductivity factor; a -- temperature conductivity factor; c -- specific heat capacity; ρ -- density; h' -- thickness of layer of turbulent liquid; λ' -- thermal conductivity factor of turbulent liquid; c' -- specific heat capacity of turbulent liquid; ρ' -- density of turbulent liquid; h_t -- thickness of layer of temperature resistance; h_S -- thickness of layer of heat capacity resistance; $\eta \equiv \frac{x}{h}$ -- relative coordinate; $\eta_0 \equiv \frac{h}{h_S}$ -- relative thickness of plate; $Fo \equiv \frac{a\tau}{h^2}$ -- Fourier number;

$\vartheta \equiv \dfrac{\lambda (t - t_0)}{Sh}$ -- relative temperature at S = const of plate surface;

$\vartheta \equiv \dfrac{t_{x=h} - t_0}{\vartheta - t_0}$ -- relative surface temperature of plate x = h when at

the other surface ϑ = const; $\rightarrow$ a symbol designating: if A is true, then B is true; BC -- boundary condition.

Introduction

Solutions to problems in thermal technology form an integral part of most areas of hydrotechnology. In some cases the results of thermal calculations serve as the initial data for solving non-thermal problems, for instance determining temperature stresses in facilities, or predicting ice conditions; in other cases knowledge of water temperature is of independent significance, for instance when water is used for industrial purposes or as a raw material, in alluvium and in concrete work, in pipe cooling as a coolant, for the purpose of irrigation, for developing the fishing industry, etc. The role, scope and complexity of the thermal problems to be examined are constantly increasing in connection with development of hydraulic construction: the implementation of related engineering solutions in erecting large facilities in difficult terrain. Various methods, especially the analytical one, of finite differences and simulation have been used to solve thermal problems. In this process, regardless of the method, general principles of physics are frequently used: symmetry, superposition, etc. The goal of the present article is to demonstrate the possibilities for applying the reciprocity principle to solving thermal conductivity problems.

The reciprocity principle was stated by Maxwell in 1864 with regard to the deformation of elastic bodies and was published in his article "Calculating the Equilibrium and Rigidity of Frames ([1], p. 598)", the essence of this principle can be stated as follows (Figure 1): "...in any linear elastic system under static load the shift δ_{BA} in the direction of one force B caused by another quantitatively equal force A is respectively equal to the shift δ_{AB} in the direction of the

-1-

second force caused by the first" ([2], p. 123). Consequently, the reciprocity principle was expanded to cover other systems as well.

Figure 1. Manifestation of the Reciprocity Principle in the Loading of an Elastic Beam.

As applied to linear electrical systems, Maxwell's reciprocity principle states that if in one section of a complex circuit electromotive force [EMF] E acts and if in the second section there appears current I, then if we transfer EMF E to the second section, in the first section there will appear current I ([3], p. 214). In general terms we can state: if in element a of a complex system excitation F acts which causes a response (reaction) H in another element of this system b, then if we transfer excitation F to element b, in element a it will cause the same response H. It is important to note that in the other system elements the responses will be different in the two cases; "reciprocity" occurs only between two selected elements. Thus, in the two cases the system is in a different condition.

In this article we will demonstrate that the reciprocity principle can also be applied to solving several problems in thermal conductivity; this means that if heat source I_S located at point 1 causes at point 2 a temperature change $\Delta t = f(\tau)$, then if we transfer the source to point 2 at point 1 the same temperature change Δt will also occur.

It should be emphasized that at the respective points the temperature change speeds are the same but the temperature gradients differ, and therefore we must remember that the shift to a reciprocal problem is not a shift to an equivalent problem: temperature fields are different.

1. The Reciprocity Principle in the Action of a Heat Source in a Semilimited Object

Let us first examine an unbounded object. If at point 1 of an unbounded isotropic and homogeneous body there acts a heat source of power I_S which causes temperature change $\Delta t = f(\tau)$ at point 2, then it is obvious that transferring heat source I_S to point 2 will cause the same temperature change $\Delta t = f(\tau)$ at point 1. The temperature fields which arise in both cases are interrelated by first-order symmetry relative to a plane which is perpendicular to a straight line connecting the two points and which intersects it in the middle. This case, of course, is trivial.

Let us make the problem more complex. Assume that points 1 and 2 are located in a semi-limited object and that an *adiabatic condition*

is assigned at the surface of the body. It is easy to see that in this case the reciprocity principle is also observed. For this purpose it is necessary to use the symmetry principle and to shift to an unlimited body with a pair of like-sign sources located within it (Figure 2). At these points 1 or 2 temperature will change under the influence of the two sources. When shifting from the initial system to the reciprocal system, the distances between the sources and the point under examination remain unchanged, and therefore the values of Δt will be the same, i.e., the reciprocity principle is observed. However, in contrast to the infinite range temperature symmetry does not occur here.
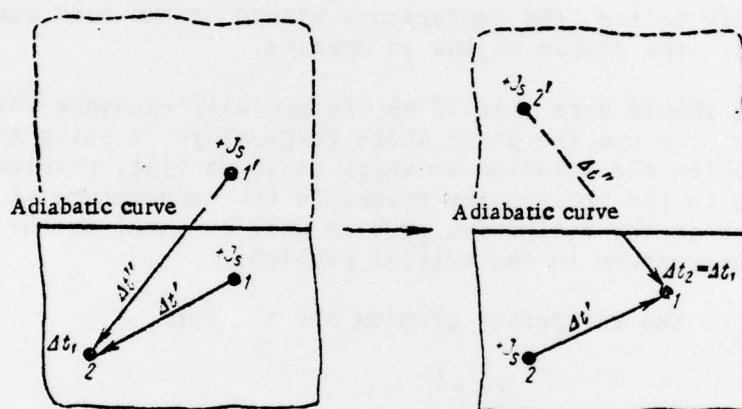
Figure 2. Reciprocity Principle When Two Heat Sources Act in a Semilimited Body: In the Initial Task the Heat Source is at Point 1; at Point 2 Occurs the Temperature Increase $\Delta t_1 = \Delta t' + \Delta t''$; In the Reciprocal Problem the Heat Source is Located at Point 2; at Point 1 the Temperature Increases is Also Equal to Δt_1.

It is important to note that if the boundary condition is an isothermic curve rather than an adiabatic curve, then the reciprocity principle is also correct; it is easy to see this if we change the sign of the sources acting at points 1' and 2' in Figure 2.

It is useful to keep in mind the following three consequences which flow from this (in this the superposition principle is also used).

In the first place, knowing the temperature change at point 2 when a single heat source acts at point 1 makes it possible to evaluate the temperature changes at point 2 when there are different values for the sources at point 1, as well as to evaluate temperature change at point 1 when a heat source of any given intensity acts at point 2.

In the second place, knowing the temperature change at points 2, 3, ..., k under the action of a single heat source at point 1 makes it possible to determine the temperature change at point 1 when heat sources act at points 2, 3, ..., k.

In the third place, conclusions on reciprocity made in the case where a point source acts can also be applied without change to cases where a linear or planar, uniformly distributed source parallel to the surface of the semilimited body acts.

Example 1. The following is given: if a semilimited body (λ = 380 Wt/m·deg, a = 0.4 m^2/hr) at a distance x = 0.5 m from the surface there acts a planar heat source S = 1000 Wt/m^2. The initial temperature is the same throughout the body. The surface body is heat-insulated.

It is necessary to find the temperature change on the body surface at τ = 5 hours after the source begins to operate.

Solution. We should note that if we reciprocally exchange the positions of the source and the plane whose temperature is being sought, then we find a problem the solution to which is known ([5], problem no. 2). According to the reciprocity principle the temperature at the plane x = 0.5 m in the reciprocal problem will be equal to the unknown surface temperature in the initial problem.

The solution to the reciprocal problem has the form

$$\Delta t = \theta \frac{S x}{\lambda}.$$

Since

$$Fo_x = \frac{a\tau}{x^2} = \frac{0.4 \cdot 5}{0.25} = 8,$$

according to the computation graph ([5], page 107) we find θ = 2.3, and consequently the desired temperature value is equal to:

$$\Delta t = 2.3 \frac{1000 \cdot 0.5}{380} \approx 3.03°C.$$

2. Reciprocity Principle When Heat Sources Act in an Unlimited Plate

The reciprocity principle is also correct for an unlimited plate on whose surface first-order or second-order BC are assigned. The correctness of this proposition is proved in essentially the same way as was the action of the reciprocity principle in the semilimited body. It is especially useful to note the existence of the reciprocity principle between the surfaces x = 0 or x = h and any other plane x (see Figure 3, a); this makes it possible to utilize the known solutions to problems with second-order BC to calculate the temperature change at the surface in the presence of internal heat sources.

Let us examine the following case in order to illustrate the correctness of the reciprocity principle for plates.

There is an unlimited plate. The boundary conditions are (Figure 4):

when x = 0

$$-\lambda \frac{\partial t}{\partial x} = S, \tag{1}$$

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when x = h

$$\frac{\partial t}{\partial x} = 0. \tag{2}$$

According to a known solution ([4], page 115; [5], p. 266; [6], page 155), the change in temperature is determined by the equation

$$\Delta t = \frac{Sh}{\lambda}\theta, \tag{3}$$

where
$$\theta = Fo - \tau_1 + \frac{\tau_1^2}{2} + \frac{1}{3} - \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{\mu_n^2} \cos[\mu_n(1-\tau_1)]\exp(-\mu_n^2 Fo), \tag{4}$$

$$\mu_n = n\pi. \tag{5}$$

Let us turn our attention to the temperature change in plane x = h/2; according to (3) it is equal to

$$\Delta t_1 = \frac{Sh}{\lambda}\theta_1; \tag{6}$$

here in determining θ_1

$$Fo_1 = \frac{a\tau}{h^2}, \tag{7}$$

$$\tau_{11} = \frac{x}{h} = \frac{1}{2}. \tag{8}$$

Now let us transfer heat source S from the surface into plane x = h/2 (Figure 4). Then, if the reciprocity principle is correct, at the boundary where the heat source previously acted temperature change Δt_2 should be equal to Δt_1. In view of the symmetric nature of the temperature field relative to plane x = h/2, the problem of determining Δt_2 can be solved by examining only one half of the plate; then it is obvious that

$$\Delta t_2 = \frac{\frac{S}{2}\frac{h}{2}}{\lambda}\theta_2, \tag{9}$$

where in determining the values of θ_2 it is necessary to assume

$$Fo_2 = \frac{a\tau}{\left(\frac{h}{2}\right)^2}, \tag{10}$$

$$\tau_{12} = 1. $$

By comparing (6) and (9), we see that $\Delta t_2 = \Delta t_1$ if
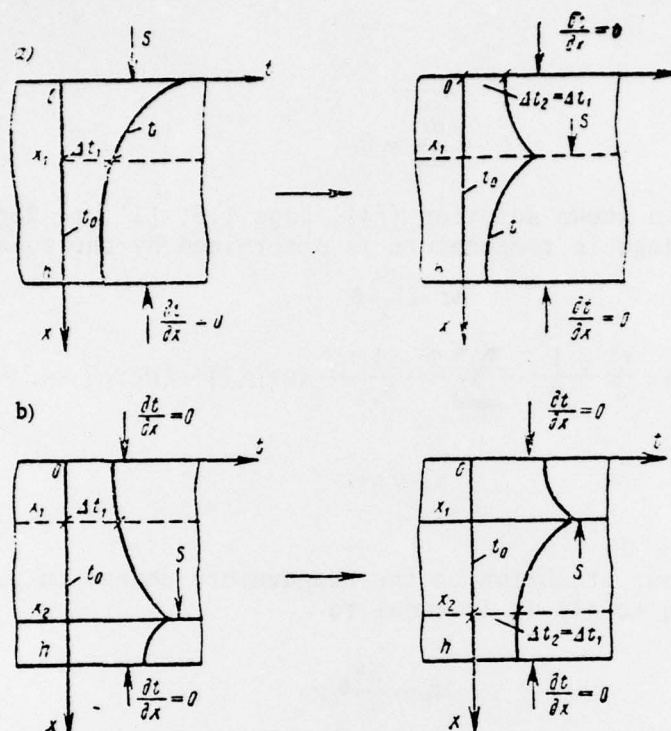
$$\theta_2 = 4\theta_1. \tag{11}$$

Figure 3. Reciprocity Principle when a Heat Source Acts in an Unlimited Plate. a, In the initial problem the heat source is located on surface $x = 0$; in the reciprocal problem the heat source is loacted in plane $x = x_1$; b, the heat source is located within the plane.

It is easy to see that condition (11) is observed; thus when $Fo > 0.5$ and when the value of the sum of the series is negligibly small, according to (4) we have

$$\vartheta_1 = Fo_1 - \frac{1}{2} + \frac{1}{8} - \frac{1}{3} = Fo_1 - \frac{1}{24}$$

and

$$\vartheta_2 = Fo_2 - 1 + \frac{1}{2} - \frac{1}{3} = Fo_2 - \frac{1}{6} = 4\left(Fo_1 - \frac{1}{24}\right).$$

Equality (11) also occurs when $Fo \leqslant 0.5$.

Thus, we have not only confirmed the correctness of the reciprocity principle but have also incidentally established the curious fact that temperature in the middle plane of the plate, at a given heat flow on one surface and with an adiabatic condition on the other surface, is equal to the temperature of the adiabatic surface of the plate if its thickness and the given heat flow are half as large (Figure 4).
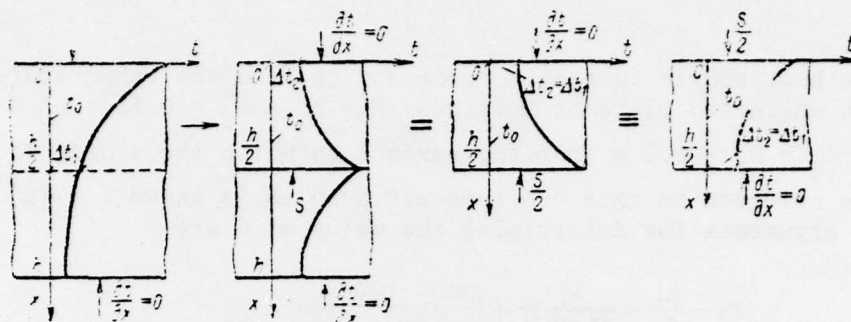
Figure 4. Proof of the Reciprocity Principle in a Plate.

Of course, the reciprocity principle also holds between two horizontal planes, parallel straight lines and individual points located within the plate (Figure 3, b).

This conclusion can also be reached by another route. Thus, for example, let us take the solution to the problem concerning the temperature curve at depth $x = x_1$ of a heat-insulated plate in which when $\tau = 0$, in plane $x = x_2$ an instantaneous single heat source acts ([4], page 354):

$$\vartheta = 1 + 2 \sum_{n=1}^{\infty} \exp(-\mu_n^2 \mathrm{Fo}) \cos \mu_n \gamma_{i1} \cos \mu_n \gamma_{i2}, \qquad (12)$$

where

$$\vartheta = \frac{t}{t_{av}}; \quad \mu_n = n\pi; \quad \gamma_{i1} = \frac{x_1}{h}; \quad \gamma_{i2} = \frac{x_2}{h}.$$

From formula (12) it follows that the temperature course is the same when the source is located in x_1 and the temperature curve is monitored in plane x_2 or whether the source is located in plane x_2 and the temperature course is monitored in plane x_1. The superposition principle makes it possible to extend this conclusion to heat sources which act constantly and are of constant or variable intensity.

Example 2. The following are given: two concrete slabs ($\lambda = 1.2$ Wt/m·deg, a = $2.57 \cdot 10^{-3}$ m^2/hr) of thickness h_1 = 0.2 m and h_2 = 0.35 m are laid one on the other; an electrical heating element (planar heat source) of power S = 500 Wt/m^2 is arranged along the surface between them. Both free surfaces of the two-plate slab are heat-insulated.

It is necessary to find the temperature change on the heat-insulated surfaces after ten hours.

Solution: a diagram of the problem solution is given in Figure 3, a.

In order to determine the temperature change on the heat-insulated surface of the first slab (of thickness h_1), it is necessary to mentally

transfer the heat source to this surface and to find the temperature change in an unlimited plate of thickness $h = h_1 + h_2 = 0.55$ m at a distance of $x_1 = h_1 = 0.2$ m from the surface on which the source is placed. The solution to this "reciprocal" problem is known [4]-[6]. The initial arguments for determining the value of θ are:

$$Fo = \frac{a\tau}{h^2} = \frac{a\tau}{(h_1 + h_2)^2} = \frac{2.57 \cdot 10^{-3} \cdot 10}{0.303} = 0.085.$$

$$\tau_t = \frac{x}{h} = \frac{h_1}{h_1 + h_2} = \frac{0.2}{0.55} = 0.36.$$

By using the computation graphs ([5], p. 145), we find $\theta = 0.1$; consequently, the temperature change is equal to

$$\Delta t = 0.1 \frac{500 \cdot 0.55}{1.2} = 22.9°C.$$

The way in which the temperature change on the second heat-insulated surface is found is similar to the one examined above. The only difference is that in this case in the reciprocal problem the heat source is transferred to another free surface; the value Fo remains as before, but the relative coordinate of the point for which the temperature change is being sought is equal to

$$\tau_t = \frac{h_2}{h_1 + h_2} = \frac{0.35}{0.55} = 0.64.$$

Having determined from the computation graph that $\theta = 0.02$, we find:

$$\Delta t = 0.02 \frac{500 \cdot 0.55}{1.2} = 4.6°C.$$

Consequently, according to the reciprocity principle the unknown temperature change on the heat-insulated surface of the first plate is equal to $\Delta t = 22.9°C$, and that on the surface of the second plate is equal to $\Delta t = 4.6°C$.

3. Reciprocity Principle when Heat Sources Act in an Unlimited Plate Covered with a Layer of Turbulent Liquid

The reciprocity principle is also valid when at the surface there exists a layer with a very high thermal conductivity factor ($\lambda' = \infty$). This assertion is by no means obvious; therefore let us examine this case in greater detail (Figure 5).

Assume, for example, that at surface x_0 of an unlimited plate h thick there is a layer of fairly agitated liquid; the initial temperature is equal to t_0, a heat flow S enters the liquid from outside and at surface $x = h$ there is an adiabatic condition (Figure 5):

$$-\lambda \frac{\partial t}{\partial x}\bigg|_{x=0} = S - c\rho h_5 \frac{\partial t}{\partial \tau}\bigg|_{x=0}, \tag{13}$$

$$\frac{\partial t}{\partial x}\bigg|_{x=h} = 0, \tag{14}$$

where

$$h_S = \frac{c'\rho'h'}{c\rho}. \tag{15}$$

The solution to this problem (we will refer to it as problem A) for the case S = const is known ([4], p. 129; [5], p. 271):

$$\theta_A = \frac{\eta_0}{\eta_0 + 1}\left[Fo + \frac{(1-\eta)^2}{2} - \frac{3+\eta_0}{6(1+\eta_0)}\right] - \tag{16}$$

$$- \sum_{n=1}^{\infty} A_n \cos[\nu_n(1-\eta)] \exp(-\nu_n^2 Fo);$$

$$\mathrm{tg}\,\nu_n = -\frac{1}{\eta_0}\,\nu_n; \tag{17}$$

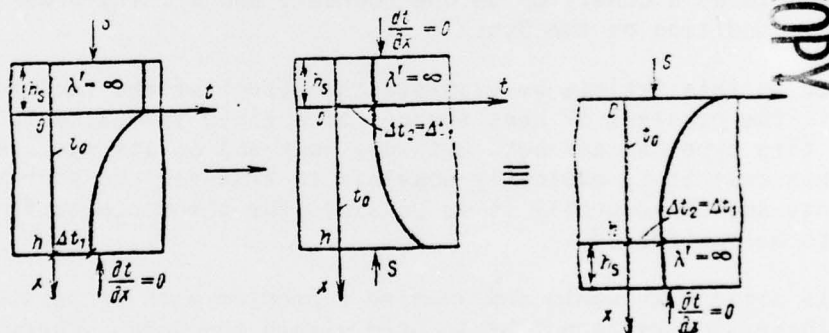$$A_n = \frac{2\eta_0^2}{\nu_n^2(\nu_n^2 + \eta_0^2 + \eta_0)\cos\nu_n}. \tag{18}$$

Figure 5. Reciprocity Principle in an Unlimited Plate with a Layer of Liquid Under the Influence of a Heat Source.

Let us now examine problem B where the heat source is located not at surface x = 0, but at surface x = h. In this case instead of (13) and (14), we have to write

$$-\lambda\frac{\partial t}{\partial x}\bigg|_{x=0} = -c\rho h_S\frac{\partial t}{\partial\tau}\bigg|_{x=0}, \tag{19}$$

$$\lambda\frac{\partial t}{\partial x}\bigg|_{x=h} = S. \tag{20}$$

The solution to this problem is also known ([4], p. 129; [5], p. 271):

$$\theta_B \equiv \frac{\eta_{l0}}{1 + \eta_0}\left[Fo \div \frac{\eta_l^2}{2} - \frac{3 \div \eta_{l0}}{6(1 + \eta_0)}\right] \div \frac{\eta_l}{1 - \eta_{l0}} -$$

$$- \sum_{n=1}^{\infty} A_n \cos\left[\nu_n(1 - \eta)\right] \exp\left(-\nu_n^2 Fo\right); \qquad (21)$$

$$A_n = \frac{2(\nu_n^2 - \eta_{l0}^2)}{\nu_n^2(\nu_n^2 + \eta_{l0}^2 + \eta_{l0})}; \qquad (22)$$

the values ν_n are determined from equation (17).

It is easy to see that if we substitute $\eta = 1$ into (16) and $\eta = 0$ into (21), then we obtain the identity

$$\theta_{A, \eta=1} = \theta_{B, \eta=0}.$$

Thus, the temperature course at the boundaries which are opposite to the boundaries with sources is identical, which indicates that the reciprocity principle is observed. In other words, transferring the layer of liquid to the opposite boundary does not change the temperature course at the boundary opposite the source.

In conclusion let us recall that the superposition principle makes it possible to extend these results to cases where the value of heat source S changes in time in accordance with any law.

4. The Reciprocity Principle in a Plate When There is a Liquid Layer and the Adiabatic Condition on One Boundary and a Third-Order Boundary Condition on the Other

Earlier in this article we examined the effect of the reciprocity principle in the presence of heat sources of a given intensity (power). Sources of this type can act both within a body and on its surface, i.e., in this case it is basically possible to transfer the sources to any points and consequently it is possible for the reciprocity principle to be manifested.

In this article we would now examine a problem with temperature sources. These sources cannot be located within the body. Therefore the possibilities for transferring the heat source are extremely limited, and the reciprocity phenomenon occurs, as you will see, somewhat differently than in the case where heat flow sources act.

We will examine the thermal regime of a plate where at surface $x = 0$ the ambient temperature (temperature source) and heat emission factor are assigned, and at a second surface $x = h$ there is a layer of agitated liquid on whose free surface the adiabatic condition is assigned, i.e., the following BC obtain (Figure 6, a):

when $x = 0$

$$- \lambda \frac{\partial t}{\partial x} = \alpha(t - t_{c.a}). \qquad (23)$$

when $x = h$

$$- \lambda \frac{\partial t}{\partial x} = c\rho h_s \frac{\partial t}{\partial \tau}. \qquad (24)$$

The solution for the case where when $\tau = 0$, $t = t_0$ and when $\tau > 0$ $\vartheta = $ const, is known ([6], p. 378; [7], p. 71). For the sake of analytical convenience we will present the function for the temperature of surface $x = h$, i.e., for the surface which is opposite to that at which the temperature source acts, as follows:

$$\vartheta \equiv \frac{t_{x=h} - t_0}{\vartheta - t} = f(\text{Fo}, \text{Bi}_{equ}). \tag{25}$$

where

$$\text{Bi}_{equ} \equiv \frac{h}{h_\vartheta}. \tag{26}$$

$$h_R = h_t - h_S - \frac{h_t h_s}{h}, \tag{27}$$

$$h_t = \frac{t}{\gamma}. \tag{28}$$

An examination of (25)-(28) shows that the temperature change at boundary $x = h$ is the same if $h_R = $ idem. This condition can be observed at various combinations of h_S and h_t values at a given plate thickness h. Therefore it is possible to draw certain conclusions concerning the possibility of replacing one problem with another in which the course of the temperature $t_x = h$ remains the same.

Thus, for instance, it is possible to switch from a problem with third-order BC on one surface and the adiabatic condition without a liquid layer on the second surface ($h_t \neq 0$, $h_S = 0$) to a problem with first-order BC and a liquid layer ($h_t = 0$, $h_S \neq 0$), based on the relationship (Figure 6, b):

$$h_S = h_t. \tag{29}$$

This same relationship (29) makes it possible to make the reverse transition from the problem with the first-order BC and the liquid layer to the problem with the third-order BC without the liquid layer.

The physical essence of relationship (29) consists of the following. In transferring a temperature source to another surface of the plate, the heat-capacity resistance should be replaced by the temperature resistance, and vice versa. In doing so the reciprocity principle is observed: the course of the temperature at the boundary opposite to that where the heat source is located will remain unchanged. Of course, with the same justification we may speak of transferring heat resistance from one surface to another with the heat source remaining in the same place. With this type of physical interpretation there arises the problem of whether it is proper to call this phenomenon a reciprocity phenomena. In answering this question in the affirmative, we consider the fact that the "responses" are unchanged at a certain point and only at this point when one of the problem's similarity conditions is changed: heat resistance at the boundaries.
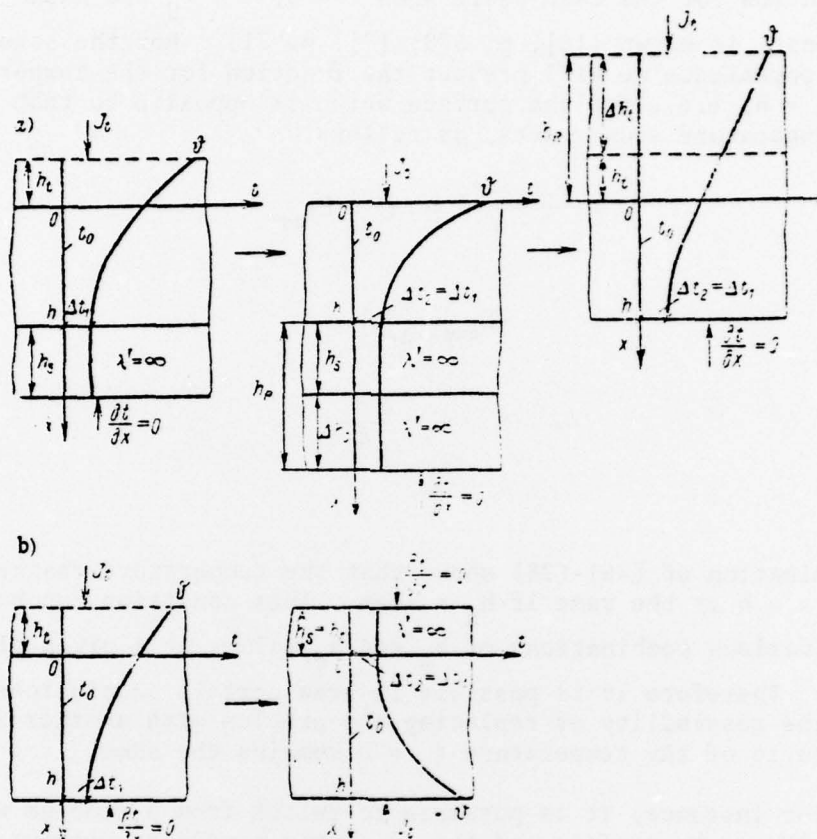
Figure 6. Reciprocity Principle in Unbounded Plate with a
Layer of Liquid and a Temperature Source, a, Transition from
problem with third-order BC and liquid layer to problem with
first-order BC and liquid layer or to problem with third-order
BC without liquid layer; b, transition from problem with
first-order BC and liquid layer to problem with third-order BC
without liquid layer.

If in the initial condition there occur third-order BC ($h_t \neq 1$),
then when there is a liquid layer at the other surface the transition
to first-order BC is carried out by increasing the thickness of the
liquid layer (Figure 6, a):

$$\Delta h_S = h_t \left(1 + \frac{h}{h_s} \right). \tag{30}$$

On the other hand, switching to the problem without the liquid
layer is accomplished by increasing the thickness of the temperature
resistance layer:

$$\Delta h_t = h_s \left(1 + \frac{h_t}{h} \right). \tag{31}$$

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We should note that in both cases, in switching from one layer (type) of heat resistance the thickness of the latter remains unchanged and is equal to h_R.

Observance of the condition h_R = idem requires that when the layer of thickness h_t is frequently changed (by Δh_t), h_S was changed by the value

$$\Delta h_S = -\Delta h_t \frac{h - h_S}{h + h_t + \Delta h_t}. \tag{32}$$

When the thickness of the liquid layer h_S is changed by the value Δh_S, it is necessary to replace h_t with the value

$$\Delta h_t = -\Delta h_S \frac{h - h_t}{h + h_S - \Delta h_S}. \tag{33}$$

Of course, these conditions are also correct for cases where the temperature source is not constant, but depends on time ($\vartheta = f(\tau)$). However, it is necessary to avoid the following false conclusion as if the solution to the problem remained the same in the case where h_S and h_t change in time but the value of h_R remains the same.

It is also interesting to note that in recent years computation functions corresponding to equation (25) have been calculated and constructed in the USA (see [7], [8]). These works have attracted considerable attention. However, in light of what we have said above it becomes clear that these data are not required since they simply repeat the long-known and tabulated solution for a plate with third-order BC without a liquid layer in which Bi = Bi_{equ} and, consequently

$$z = z_{equ} = \frac{h}{h_s}. \tag{34}$$

It is easy to see that the temperatures of the adiabatic surface are identical in both problems if we compare the graphs of 23f and 24b with 30b in [7] or graphs 6.10 and 6.11 with 10.5 in [6]; it is only necessary to keep in mind the fact that in the graphs for a plate with liquid in [6] and [7] the parameter μ corresponds to the values of $1/Bi_{equ}$ in this case.[1]

Example 3. The following are given: a concrete slab ($\lambda = 1.2$ Wt/m·deg, c = 0.233 Wt/kg·deg, $\rho = 2000$ kg/m^3, a = $2.57 \cdot 10^{-3}$ m^2/hr) of thickness h = 0.5 m is heated in a gas medium with $\vartheta = 300°C$ and a heat emission factor of $\alpha = 30$ Wt/m^2·deg. The initial temperature of the slab is $t_0 = 10°C$. The lower surface of the slab makes contact with a layer of agitated water (c' = 1.163 Wt·hr/kg·deg; $\rho' = 1000$ kg/m^2; $\lambda' = \infty$) of thickness h' = 0.25 m.

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[1] In [6] the captions under Figures 10.4 and 10.5 are transposed.

It is necessary to find how long it takes the water to be heated up to t = 90°C.

Solution: in accordance with (27) we find

$$h_R = h_t + h_s + \frac{h_t h_s}{h} = \frac{\lambda}{\alpha} + \frac{c'\rho'h'}{c\rho} + \frac{\lambda}{\alpha} \frac{c'\rho'h'}{c\rho h} =$$

$$= \frac{1.2}{30} + \frac{1,163 \cdot 1000 \cdot 0.25}{0.233 \cdot 2000} + \frac{1.2}{30} \frac{1,163 \cdot 1000 \cdot 0.25}{0.233 \cdot 2000 \cdot 0.5} = 0.715 \; м.$$

from which according to (26)

$$\mathrm{Bi}_{equ} = \frac{h}{h_R} = \frac{0.5}{0.715} = 0.7.$$

According to (25)

$$\theta = \frac{t_{x=h} - t_o}{\vartheta - t_o} = f(\mathrm{Fo}, \mathrm{Bi}_{equ}).$$

$$\theta = \frac{90 - 10}{300 - 10} = 0.276.$$

By utilizing the graph for computing the temperature course ([6], p. 203) for η = 1, we find that when θ = 0.276, Fo = 0.73. From this we find the desired value

$$\tau = \mathrm{Fo} \frac{h^2}{a} = 0.73 \frac{0.25}{2.57 \cdot 10^{-3}} = 71 \text{ hrs.}$$

Example 4. The following are given: the conditions of example 3, with the exception of the boundary condition: at the boundary there obtains a linear law of temperature change $\vartheta = t_0{}'b\tau$. The temperature of the gas medium increases with a constant speed b = 4.0 deg/hr. The initial temperature of the concrete plate and the gas medium is the same $t_0 = \vartheta = 10°$C.

It is necessary to find how long it takes the water to be heated to t = 90°C.

Solution. In order to solve this problem we utilize the computation graph ([5], p. 192). The initial arguments of the problem are

$$\mathrm{Bi}_{equ} = 0.7 \text{ and } \theta = \frac{a(t - t_o)}{bh^2} = \frac{2.57 \cdot 10^{-3} \cdot 80}{4.0 \cdot 0.25} = 0.206;$$

when η = 1 and θ = 0.206 and Bi_{equ} = 0.7, we find Fo = 1.1 and consequently

$$\tau = \mathrm{Fo} \frac{h^2}{a} = 1.1 \frac{0.25}{2.57 \cdot 10^{-3}} \approx 107 \text{ hrs.}$$

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Conclusions

This article has shown that the phenomenon expressed as the reciprocity principle which is known in various branches of physics also occurs when heat propagates in solids. This conclusions is not only of purely scientific interests. Thus, utilization of the reciprocity principle makes it possible to find analytical solutions for certain thermal problems, the solution to which has not yet been obtained by means of other methods. In addition, it should prove useful to apply the reciprocity principle in setting up laboratory and whole-scale studies since for practical purposes in a number of cases it is convenient to transpose the locations of the heat sources and the points in the object the temperature of which has to be measured. The possibility of exploiting the reciprocity principle in design is no less important. Thus, for instance function (32) indicates how to replace heat-insulation material at surface $x = 0$ either partially or completely with a layer of liquid at surface $x = h$ so that the heat-protection effect of surface $x = h$ remains unchanged.

The reciprocity phenomenon has been studied in this article only with regard to problems in non-stationary thermal conductivity in homogeneous isotropic plates of limited and unlimited thickness and under various boundary conditions, including the presence on the body surface of a layer with a very high heat conductivity factor and with internal heat sources. There is no doubt that attempts to apply this principle to other problems, for instance rectangular, equilateral, polygonal and isosceles triangles, straight prisms based on the above--mentioned planar figures, as well as cylinders, spheres, etc., will provide positive results.

In the future it will be advisable to search for more general laws to govern the manifestation of the reciprocity principle in thermal conductivity processes. This will make it possible to expand the range of application for the reciprocity principle while at the same time delimiting its boundaries.

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